Well-balanced schemes for rotating shallow water equations.

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Introduction

2D rotating shallow water (RSW) equations :

$$\begin{cases} \partial_t h + \partial_x(hu) + \partial_y(hv) = 0, \\ \partial_t(hu) + \partial_x\left(hu^2 + \dfrac{gh^2}{2}\right) + \partial_y(huv) = -gh\partial_x z + fhv \\ \partial_t(hv) + \partial_x(huv) + \partial_y\left(hv^2 + \dfrac{gh^2}{2}\right) = -gh\partial_y z - fhu \end{cases}$$

- $h(x,t)$: fluid height
- $u(x,t), v(x,t)$: velocity
- $z(x)$: topography
- f : Coriolis force
- g : gravity constant

Introduction

1D rotating shallow water (RSW) equations :

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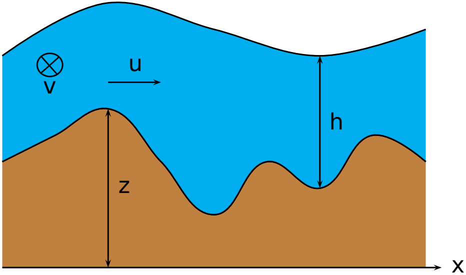

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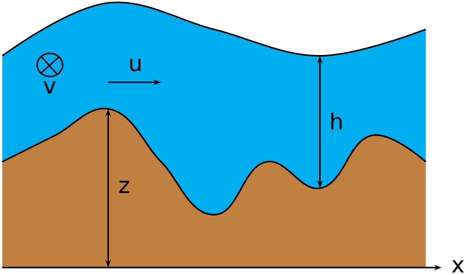

$$w = (h, hu, hv)^T, \widetilde{w} = (w, z).$$
$$\partial_t w + \partial_x f(w) = s(\widetilde{w}).$$

Goals

- Steady states preservation

$$\begin{cases} hu = q_0 \\ \partial_x \left(\dfrac{u^2}{2} + g(h+z) \right) = fv \\ q_0 \partial_x v = -f q_0 \end{cases}$$

$$\begin{cases} u = 0 \\ g\partial_x (h+z) = fv \end{cases}$$

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Lake at rest for SW equations

- Hydrostatic reconstruction :
  Audusse, Bouchut, Bristeau, Klein, and Perthame 2004
- Relaxation : Bouchut 2004
- Approximate Riemann solver : Berthon and Foucher 2012

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Moving steady states for SW equations

- Noelle, Xing, and Shu 2007
- Berthon and Chalons 2016
- Michel-Dansac, Berthon, Clain, and Foucher 2017
- Berthon, M'baye, Le, and Seck 2021

Goals

- Steady states preservation

$$\begin{cases} hu = q_0 \\ \partial_x \left(\dfrac{u^2}{2} + g(h+z) \right) = fv \\ q_0 \partial_x v = -fq_0 \end{cases}$$

$$\begin{cases} u = 0 \\ g\partial_x (h+z) = fv \end{cases}$$

Steady state at rest for RSW equations

- Bouchut, Le Sommer, and Zeitlin 2005
- Lukáčová-Medvid'ová, Noelle, and Kraft 2007,
- Chertock, Dudzinski, Kurganov, and Lukáčová-Medvid'ová 2018.

Goals

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- Positive-preserving scheme $\forall (x,t) \in \mathbb{R} \times \mathbb{R}^+, h(x,t) > 0$.
  - cut-off procedure : Audusse, Chalons, Ung 2015

Goals

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- Positive-preserving scheme $\forall (x,t) \in \mathbb{R} \times \mathbb{R}^+, h(x,t) > 0$.
  - cut-off procedure : Audusse, Chalons, Ung 2015
- High order approximation
  MUSCL method
  - Van Leer 1979
  - Michel-Dansac, Berthon, Clain, and Foucher 2016,
  - Ghitti, Berthon, Le, and Toro 2020.

Outline

1. Godunov-type schemes
2. A simple well-balanced scheme
3. A fully well-balanced scheme
4. Second-order extension

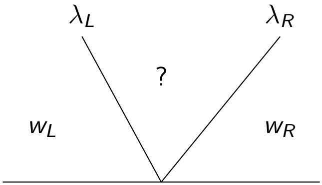
Outline

Definitions

Riemann problem :
A Cauchy problem with an initial condition like

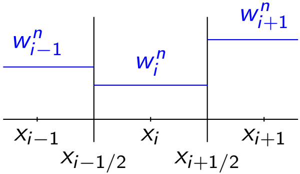
$$w(x,0) = \begin{cases} w_L & \text{if } x < 0, \\ w_R & \text{if } x > 0. \end{cases}$$

Exact Riemann solver :
Solution of a Riemann problem denoted $\mathcal{W}_R\left(\frac{x}{t}, w_L, w_R\right) \subset \mathbb{R}^3$



Godunov scheme

Finite volume discretisation at time t^n is a juxtaposition of Riemann problems



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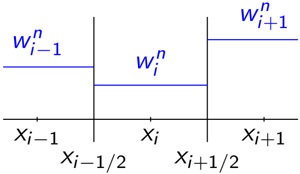
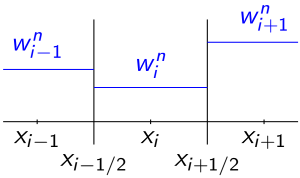


Approximation at time t^{n+1} obtained in two steps :

- Exact evolution
  $$w_{\Delta x}(x, t^n + t) = \mathcal{W}_R\left(\frac{x - x_{i+1/2}}{t}, \widetilde{w}_i^n, \widetilde{w}_{i+1}^n\right) \text{ on } [x_i, x_{i+1}].$$
- Projection
  $$w_i^{n+1} = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} w_{\Delta x}(x, t^n + \Delta t)dx.$$

Godunov-type scheme

Finite volume discretisation at time t^n is a juxtaposition of Riemann problems



Approximation at time t^{n+1} obtained in two steps :

- Approximate evolution
  $$w_{\Delta x}(x, t^n + t) = \widehat{\mathcal{W}}_R\left(\frac{x - x_{i+1/2}}{t}, \widetilde{w}_i^n, \widetilde{w}_{i+1}^n\right) \text{ on } [x_i, x_{i+1}].$$
- Projection
  $$w_i^{n+1} = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} w_{\Delta x}(x, t^n + \Delta t)dx.$$

where $\widehat{\mathcal{W}}_R\left(\frac{x}{t}, \widetilde{w}_L, \widetilde{w}_R\right)$ is an approximate Riemann solver.

Approximate Riemann solver

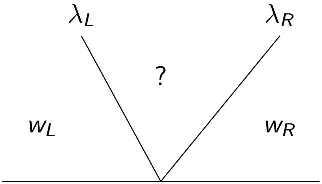


Figure: Exact Riemann solver

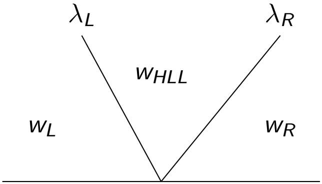


Figure: HLL approximate Riemann solver

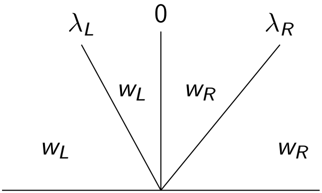


Figure: Approximate Riemann solver **at steady state**.

Consistency

$$\frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \widehat{\mathcal{W}}_R\left(\frac{x}{\Delta t}, \widetilde{w}_L, \widetilde{w}_R\right) dx = \frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \mathcal{W}_R\left(\frac{x}{\Delta t}, \widetilde{w}_L, \widetilde{w}_R\right) dx$$

Without source terms:

$$\frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \widehat{\mathcal{W}}_R\left(\frac{x}{\Delta t}, \widetilde{w}_L, \widetilde{w}_R\right) dx = \frac{w_L + w_R}{2} - \frac{\Delta t}{\Delta x}(f(w_R) - f(w_L))$$

Consistency

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With~~out~~ source terms :

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$$+\frac{1}{\Delta x}\int_0^{\Delta t}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} s\left(\mathcal{W}_R\left(\frac{x}{t}, \widetilde{w}_L, \widetilde{w}_R\right), z(x)\right) dxdt$$

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where $$S^{hu}(\widetilde{w}_L,\widetilde{w}_R) = fhv\Delta x + gh(z_R - z_L) + o((z_R - z_L),\Delta x),$$

and

$$S^{hv}(\widetilde{w}_L,\widetilde{w}_R) = -fhu\Delta x + o(\Delta x),$$

as $\widetilde{w}_L,\widetilde{w}_R \to \widetilde{w}$.

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as $\widetilde{w}_L, \widetilde{w}_R \to \widetilde{w}$.

Remark : At steady state, we want

$$S(\widetilde{w}_L, \widetilde{w}_R) = f(w_R) - f(w_L).$$

Steady states discretisation

Let (w_L, w_R) be a Riemann problem initial data.

$[X] = X_R - X_L$, $\bar{X} = \frac{X_R + X_L}{2}$.

- Steady states at rest

$$\begin{cases} u = 0, \\ g\partial_x (h + z) = fv, \end{cases} \qquad \text{(WB)} \begin{cases} u_L = u_R = 0, \\ g\,[h + z] = \Delta x f \overline{v}, \end{cases}$$

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- All steady states

$$\begin{cases} hu = q_0, \\ \partial_x \left(\dfrac{u^2}{2} + g(h+z) \right) = fv, \\ q_0 \partial_x v = -f q_0. \end{cases} \qquad \text{(FWB)} \begin{cases} h_R u_R = h_L u_L = q_0, \\ \left[\dfrac{u^2}{2} + g(h+z) \right] = \Delta x f \overline{v}, \\ q_0 [v] = -\Delta x f q_0. \end{cases}$$

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The couple (w_L, w_R) is at steady state if it satisfies the relations (WB) or (FWB).

Outline

WB approximate Riemann solver

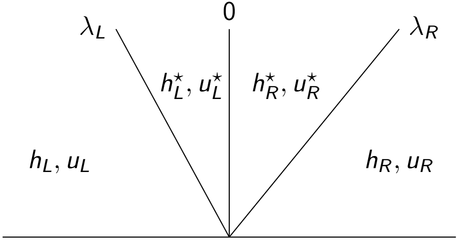
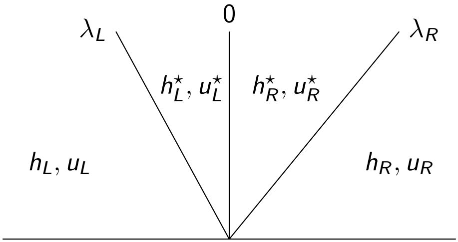


Figure: WB approximate Riemann solver $\widehat{\mathcal{W}}_{WB}$

WB approximate Riemann solver



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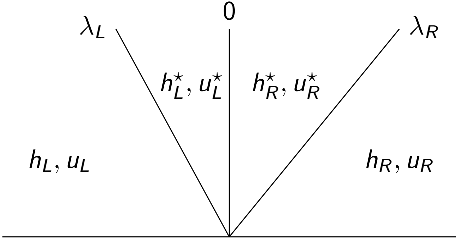
- Consistency in h and hu

WB approximate Riemann solver



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- Consistency in h and hu
- WB relations :

$$0 = [hu] \Rightarrow q^\star = h_R^\star u_R^\star = h_L^\star u_L^\star,$$

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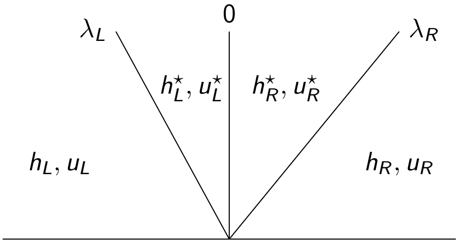

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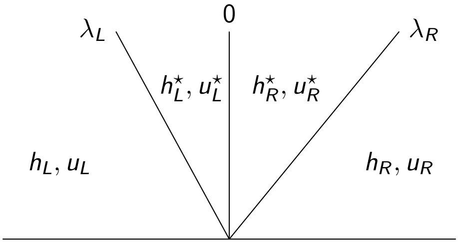

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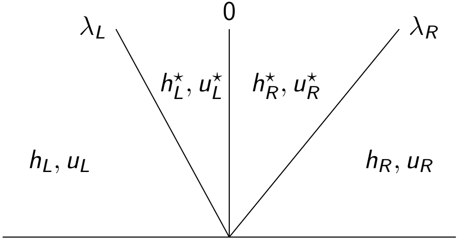
$$\Rightarrow h_R^\star - h_L^\star = S^{hu}/(g\overline{h}).$$

WB approximate Riemann solver



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$$\Rightarrow h_R^\star - h_L^\star = S^{hu}/(g\bar{h}).$$

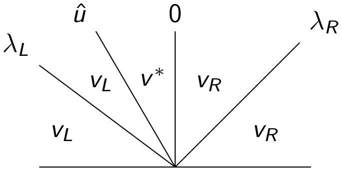
$$q^\star = q_{HLL} + \frac{S^{hu}}{\lambda_R - \lambda_L},$$

$$h_{L,R}^\star = h_{HLL} - \frac{\lambda_{R,L}}{\lambda_R - \lambda_L}\frac{S^{hu}}{g\bar{h}}.$$

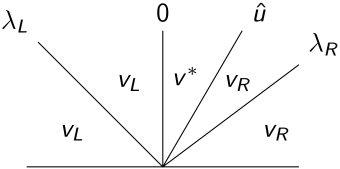
WB approximate Riemann solver

$$\partial_t(hv) + \partial_x(huv) = -fhu$$

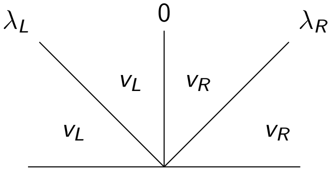
$$\partial_t v + u\partial_x v = -fu$$



(a) $\hat{u} < 0$



(b) $\hat{u} > 0$



(c) $\hat{u} = 0$

WB approximate Riemann solver

Definition of $v^\star$:

$$v^\star = \begin{cases} v_L - f\Delta x \text{ si } \hat{u} > 0, \\ v_R + f\Delta x \text{ si } \hat{u} < 0. \end{cases}$$

Definition of $\hat{u}$: It comes from the consistency in hv. We set $C = ([v] - f\Delta x)$.

- If $C \neq 0$ then

$$\hat{u} = \begin{cases} \frac{h_L u_L + \lambda_L(h_L^\star - h_L)}{h_R^\star} & \text{if } h_L u_L + \lambda_L(h_L^\star - h_L) > 0, \\ \frac{h_L u_L + \lambda_L(h_L^\star - h_L)}{h_L^\star} & \text{if } h_L u_L + \lambda_L(h_L^\star - h_L) < 0, \\ 0 & \text{if } h_L u_L + \lambda_L(h_L^\star - h_L) = 0, \end{cases}$$

- Else, by definition of $v^\star$, we always have

$$v(x,t) = \begin{cases} v_L \text{ if } x > 0, \\ v_R \text{ if } x < 0. \end{cases}$$

We still need to satisfy the consistency in hv in that case.

Source term discretisation

Continuous source terms

$$s^{hu}(w) = fhv - gh\partial_x z$$
$$s^{hv}(w) = -fhu$$

Steady state discretisation

$$\text{(WB)} \begin{cases} u_L = u_R = 0, \\ g\,[h+z] = \Delta x f \overline{v}, \end{cases}$$

Source term discretisation

Continuous source terms

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Steady state discretisation

$$\text{(WB)} \begin{cases} u_L = u_R = 0, \\ g\,[h+z] = \Delta x f \overline{v}, \end{cases}$$

We define S^{hu} according to the well-balanced assessment :

$$S^{hu} = [hu^2 + gh^2/2] = g\overline{h}[h] = f\,\overline{h}\overline{v}\Delta x - g\,\overline{h}[z].$$

Source term discretisation

Continuous source terms

$$s^{hu}(w) = fhv - gh\partial_x z$$
$$s^{hv}(w) = -fhu$$

Steady state discretisation

$$(\text{WB}) \begin{cases} u_L = u_R = 0, \\ g\,[h+z] = \Delta x f \overline{v}, \end{cases}$$

We define S^{hu} according to the well-balanced assessment :

$$S^{hu} = [hu^2 + gh^2/2] = g\overline{h}[h] = f\,\overline{h}\overline{v}\Delta x - g\overline{h}[z].$$

We define S^{hv} according to the consistency condition when $C = [v] - f\Delta x = 0$:

$$S^{hv} = -f\,\overline{hu}\Delta x - \frac{f}{2}\Delta x(\lambda_R(h_R^\star - h_R) + \lambda_L(h_L^\star - h_L)).$$

Outline

FWB approximate Riemann solver

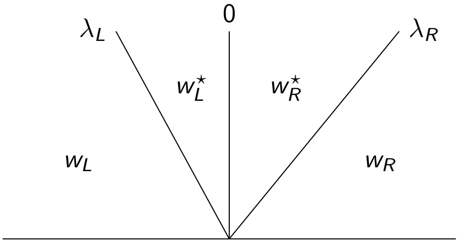
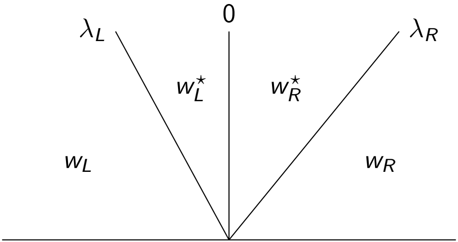


Figure: FWB approximate Riemann solver $\widehat{\mathcal{W}}_{FWB}$

6 unknowns:

- 3 relations from the consistency
- 3 others from the FWB condition

FWB approximate Riemann solver



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6 unknowns:

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Steady state indicator :

$$\mathcal{E}_{LR} = \sqrt{\left|\,[hu]\,\right|^2 + \left|\,\left[\frac{u^2}{2} + g(h+z)\right] - \Delta x f \overline{v}\right|^2 + \left|\overline{hu}([v] + f\Delta x)\right|^2}.$$

Definition of $w^\star_{L,R}$

At steady state we have

$$0 = [hu] \Longrightarrow h_L u_L = h_R u_R = q_0$$
$$S^{hu} = [hu^2 + gh^2/2] = \left(g\bar{h} - |u_L u_R|\right)(h_R - h_L)$$
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Definition of $w^\star_{L,R}$

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$$v^\star_R - v^\star_L = \begin{cases} \dfrac{\widetilde{q}\, S^{hv}}{\widetilde{q}^2 + \mathcal{E}_{LR}} & \text{if } \mathcal{E}_{LR} \neq 0 \\ v_R - v_L & \text{if } \mathcal{E}_{LR} = 0 \end{cases}$$

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Source term discretisation

Continuous source terms

$$s^{hu}(w) = fhv - gh\partial_x z$$
$$s^{hv}(w) = -fhu$$

Steady state discretisation

$$\text{(FWB)} \begin{cases} h_R u_R = h_L u_L = q_0 \\ \left[\frac{u^2}{2} + g(h+z) \right] = f \Delta x \overline{v} \\ q_0 [v] = -f \Delta x q_0 \end{cases}$$

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Using the steady state discretisation, we can write

$$h_R - h_L = \frac{\Delta x f \overline{v}/g - [z]}{1 - \mathrm{Fr}}, \text{ where } \mathrm{Fr} = \frac{\overline{h}|u_L u_R|}{g h_L h_R}$$

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becomes

$$S^{hu} = \Delta x f \overline{h}\overline{v} - g\overline{h}[z] + \frac{g\mathrm{Fr}[h]}{4\overline{h}} \frac{(\Delta x f \overline{v}/g - [z])^2}{(1 - \mathrm{Fr})^2}.$$

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Remark : $\lim_{\mathrm{Fr}\to 1} S^{hu} = \Delta x f \overline{h}\overline{v} - g\overline{h}[z] + \frac{g}{4\overline{h}}[h]^3.$

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$$\boxed{S^{hu} = \Delta x f \overline{h}\overline{v} - g\overline{h}[z] + \frac{g\mathrm{Fr}[h]}{4\overline{h}} \frac{(\Delta x f \overline{v}/g - [z])^2}{(1 - \mathrm{Fr})^2 + \mathcal{E}_{LR}}.}$$

Approximate Riemann solvers properties

- positive-preserving :
  Let $\varepsilon > 0$. We set $\delta := \min(\varepsilon, h_L, h_R, h^{HLL})$.
  If $h_L^\star < \delta$, we set $h_L^\star = \delta$ and we define $h_R^\star$ with respect to the consistency relation.
  $$h_R^\star = \left(1 - \frac{\lambda_L}{\lambda_R}\right) h^{HLL} + \frac{\lambda_L}{\lambda_R} h_L^\star \geq \delta.$$
  We proceed similarly if $h_R^\star < \delta$.

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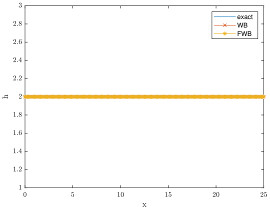
Theorem

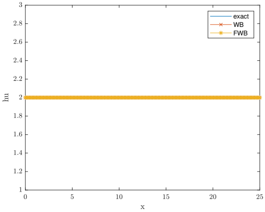
Under a 1/2-CFL restriction, the WB and FWB schemes preserve the expected steady states and preserves the positivity of h.

Numerical results : easy moving steady state

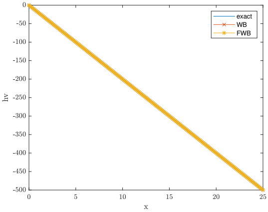
$$h_0(x) = 2, u_0(x) = 1, v_0(x) = -fx, z(x) = \frac{-f^2x^2}{2}.$$



(a) h



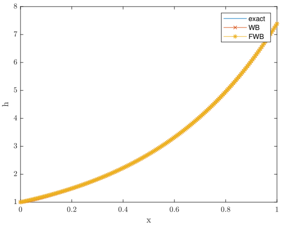
(b) hu



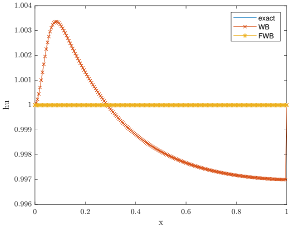
(c) hv

Numerical results : moving steady state

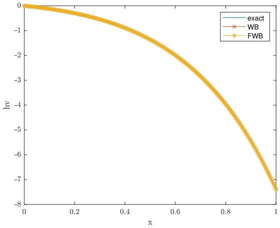
$$h_0(x) = \exp(2x), u_0(x) = \exp(-2x), v_0(x) = -fx,$$
$$z(x) = \frac{-f^2x^2}{2} - \exp(2x) - \frac{\exp(-4x)}{2}.$$



(d) h



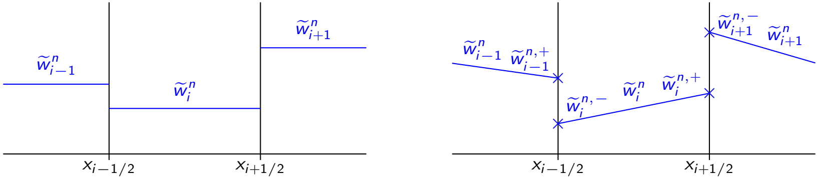
(e) hu



(f) hv

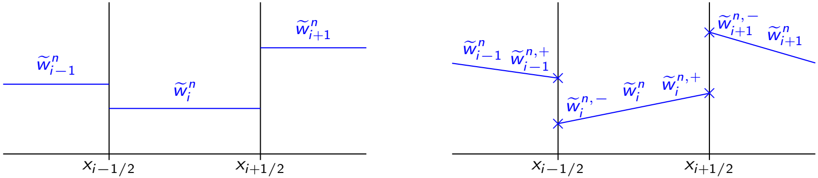
Outline

Piecewise reconstruction



$$\widetilde{w}_i^{n,\pm} = \widetilde{w}_i^n \pm \frac{\Delta x}{2}\sigma_i^n(\widetilde{w}),$$

Piecewise reconstruction

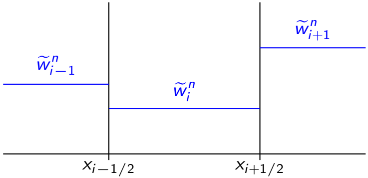


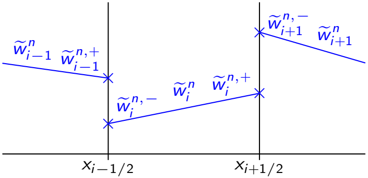
$$\widetilde{w}_i^{n,\pm} = \widetilde{w}_i^n \pm \frac{\Delta x}{2}\sigma_i^n(\widetilde{w}),$$

$$\sigma_i^n(\widetilde{w}) = \mathrm{minmod}\left(\frac{\widetilde{w}_i^n - \widetilde{w}_{i-1}^n}{\Delta x}, \frac{\widetilde{w}_{i+1}^n - \widetilde{w}_i^n}{\Delta x}\right)$$

$$\mathrm{minmod}(\sigma_L, \sigma_R) = \begin{cases} \min(\sigma_L, \sigma_R) & \text{if } \sigma_L > 0 \text{ and } \sigma_R > 0, \\ \max(\sigma_L, \sigma_R) & \text{if } \sigma_L < 0 \text{ and } \sigma_R < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Piecewise reconstruction





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Possibility to reconstruct other variables than $\widetilde{w}$, in order to get reconstruction that preserves steady states.

Reconstruction preserving steady states

Preservation of lake at rest

$$\begin{cases} u = 0, \\ h + z = cste, \end{cases}$$

Preservation of steady states at rest

$$\begin{cases} u = 0, \\ g\partial_x(h + z) = f\bar{v}, \end{cases}$$

Preservation of all steady states

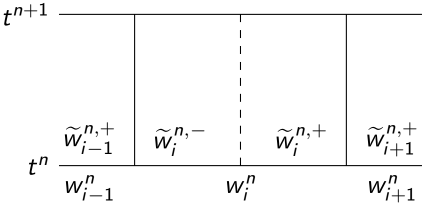
$$\begin{cases} hu = q_0, \\ \partial_x \left(\dfrac{u^2}{2} + g(h + z) \right) = f\overline{v}, \\ q_0 \partial_x v = -fq_0. \end{cases}$$

MUSCL method

First-order scheme :

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\left(F\left(\widetilde{w}_i^n, \widetilde{w}_{i+1}^n, \Delta x\right) - F\left(\widetilde{w}_{i-1}^n, \widetilde{w}_i^n, \Delta x\right)\right) + \frac{\Delta t}{2\Delta x}\left(S\left(\widetilde{w}_{i-1}^n, \widetilde{w}_i^n, \Delta x\right) + S\left(\widetilde{w}_i^n, \widetilde{w}_{i+1}^n, \Delta x\right)\right).$$

Second-order scheme :



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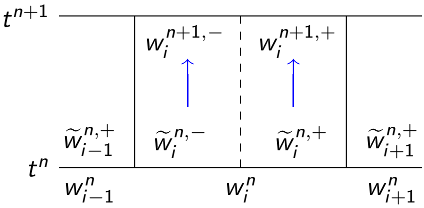

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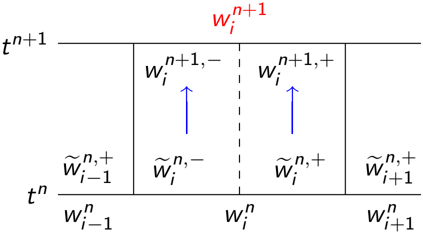

$$w_i^{n+1} = \frac{1}{2}\left(w_i^{n+1,-} + w_i^{n+1,+}\right)$$

MUSCL method

Second-order scheme :

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Required properties on reconstruction:

- conservative : $\frac{w_i^{n,+} + w_i^{n,-}}{2} = w_i^n$, reconstruction of h, hu, hv.
- (F)WB reconstruction : only one degree of freedom

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Incompatible conditions for the FWB.

Fully well-balanced recovering

Key idea :

- Consider the second-order scheme far from steady state,
- Use the first-order scheme at steady state.

New reconstruction :

$$\widetilde{w}_i^{n,\pm} = \widetilde{w}_i^n \pm \theta_i^n \frac{\Delta x}{2} \sigma_i^n(\widetilde{w}),$$

$$\theta_i^n \approx \begin{cases} 0 & \text{at steady state,} \\ 1 & \text{far from a steady state.} \end{cases}$$

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$$\text{with } \Delta x_1 = \Delta x\left(1 - \frac{\theta_i^n}{2}\right) \quad \text{and} \quad \Delta x_2 = \theta_i^n \frac{\Delta x}{2}.$$

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$$\theta_i^n \approx \begin{cases} 0 & \text{at steady state,} \\ 1 & \text{far from a steady state.} \end{cases}$$

$$\theta(x) = \frac{x^2}{x^2 + \Delta x^2}.$$

$$\theta_i^n = \theta(\mathcal{E}(\widetilde{w}_{i-1}^n, \widetilde{w}_i^n, \Delta x) + \mathcal{E}(\widetilde{w}_i^n, \widetilde{w}_{i+1}^n, \Delta x))$$

Scheme properties

$$h_i^{n+1} = h_i^n - \frac{\Delta t}{\Delta x} \left(F^h \left(\widetilde{w}_i^{n,+}, \widetilde{w}_{i+1}^{n,-}, \Delta x_1 \right) - F^h \left(\widetilde{w}_{i-1}^{n,+}, \widetilde{w}_i^{n,-}, \Delta x_1 \right) \right)$$

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&= \frac{1}{2}\left(h_i^{n,-} - \frac{\Delta t}{\Delta x/2}\left(F^h\left(\widetilde{w}_i^{n,-},\widetilde{w}_i^{n,+},\Delta x_1\right) - F^h\left(\widetilde{w}_{i-1}^{n,+},\widetilde{w}_i^{n,-},\Delta x_1\right)\right)\right)\\
&\quad + \frac{1}{2}\left(h_i^{n,+} - \frac{\Delta t}{\Delta x/2}\left(F^h\left(\widetilde{w}_i^{n,+},\widetilde{w}_{i+1}^{n,-},\Delta x_1\right) - F^h\left(\widetilde{w}_i^{n,-},\widetilde{w}_i^{n,+},\Delta x_1\right)\right)\right)
\end{aligned}
$$

Scheme properties

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\begin{aligned}
h_i^{n+1} &= h_i^n - \frac{\Delta t}{\Delta x}\left(F^h\left(\widetilde{w}_i^{n,+}, \widetilde{w}_{i+1}^{n,-}, \Delta x_1\right) - F^h\left(\widetilde{w}_{i-1}^{n,+}, \widetilde{w}_i^{n,-}, \Delta x_1\right)\right) \\
&= \frac{1}{2}\left(h_i^{n,-} - \frac{\Delta t}{\Delta x/2}\left(F^h\left(\widetilde{w}_i^{n,-}, \widetilde{w}_i^{n,+}, \Delta x_1\right) - F^h\left(\widetilde{w}_{i-1}^{n,+}, \widetilde{w}_i^{n,-}, \Delta x_1\right)\right)\right) \\
&\quad + \frac{1}{2}\left(h_i^{n,+} - \frac{\Delta t}{\Delta x/2}\left(F^h\left(\widetilde{w}_i^{n,+}, \widetilde{w}_{i+1}^{n,-}, \Delta x_1\right) - F^h\left(\widetilde{w}_i^{n,-}, \widetilde{w}_i^{n,+}, \Delta x_1\right)\right)\right) \\
&\geq 0,
\end{aligned}
$$

since the cut-off procedure does not depend on Δx_1.

Scheme properties

$$
\begin{aligned}
h_i^{n+1} &= h_i^n - \frac{\Delta t}{\Delta x}\left(F^h\left(\widetilde{w}_i^{n,+}, \widetilde{w}_{i+1}^{n,-}, \Delta x_1\right) - F^h\left(\widetilde{w}_{i-1}^{n,+}, \widetilde{w}_i^{n,-}, \Delta x_1\right)\right) \\
&= \frac{1}{2}\left(h_i^{n,-} - \frac{\Delta t}{\Delta x/2}\left(F^h\left(\widetilde{w}_i^{n,-}, \widetilde{w}_i^{n,+}, \Delta x_1\right) - F^h\left(\widetilde{w}_{i-1}^{n,+}, \widetilde{w}_i^{n,-}, \Delta x_1\right)\right)\right) \\
&\quad + \frac{1}{2}\left(h_i^{n,+} - \frac{\Delta t}{\Delta x/2}\left(F^h\left(\widetilde{w}_i^{n,+}, \widetilde{w}_{i+1}^{n,-}, \Delta x_1\right) - F^h\left(\widetilde{w}_i^{n,-}, \widetilde{w}_i^{n,+}, \Delta x_1\right)\right)\right) \\
&\geq 0,
\end{aligned}
$$

since the cut-off procedure does not depend on Δx_1.

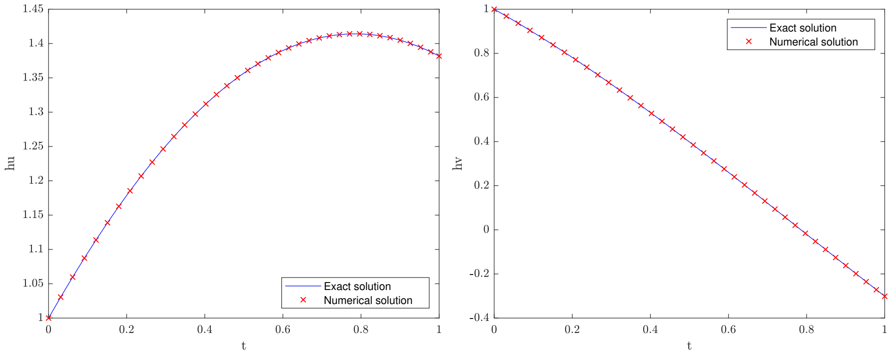
Theorem

Under a 1/4-CFL restriction, the second-order scheme is fully well-balanced and preserves the positivity of h.

Numerical results : Steady state in time

For a constant initial condition (h_0, u_0, v_0) fixed, the exact solution of RSW equations writes

$$h(x,t) = h_0,$$
$$u(t) = u_0 \cos(ft) + v_0 \sin(ft),$$
$$v(t) = v_0 \cos(ft) - u_0 \sin(ft).$$



Numerical results : Steady state in time

Table: L^1-error, first-order scheme, $T_{\max} = 1$.

N	hu		hv	
200	3.82×10^{-4}	0.99	8.06×10^{-5}	0.99
400	1.91×10^{-4}	0.99	4.03×10^{-5}	0.99
800	9.56×10^{-5}	0.99	2.01×10^{-5}	0.99
1600	4.78×10^{-5}	0.99	1.01×10^{-5}	0.99
3200	2.39×10^{-5}	0.99	5.04×10^{-6}	0.99
6400	1.20×10^{-5}	0.99	2.52×10^{-6}	0.99

Table: L^1-error, second-order scheme, $T_{\max} = 1$.

N	hu		hv	
200	7.71×10^{-9}	1.99	3.58×10^{-8}	1.99
400	1.92×10^{-9}	1.99	8.95×10^{-9}	1.99
800	4.82×10^{-10}	2.00	2.24×10^{-9}	1.99
1600	1.20×10^{-10}	2.00	5.60×10^{-10}	1.99
3200	3.01×10^{-11}	2.00	1.40×10^{-10}	1.99
6400	7.52×10^{-12}	2.00	3.50×10^{-11}	1.99

Numerical results : geostrophic steady state

$$h_0(x) = \frac{2}{g} - e^{-x^2}, u_0(x) = 0, v_0(x) = \frac{2g}{f} x e^{-x^2},$$
$$f = 10, g = 1.$$

Numerical results : geostrophic steady state

$$h_0(x) = \frac{2}{g} - e^{-x^2}, u_0(x) = 0, v_0(x) = \frac{2g}{f} x e^{-x^2},$$
$$f = 10, g = 1.$$

(c) L^1 error in space, first-order scheme, $T_{\max} = 200$

N	h		hv	
200	6.15×10^{-1}		1.89×10^{-1}	
400	4.01×10^{-1}	0.62	1.35×10^{-1}	0.48
800	2.43×10^{-1}	0.72	8.81×10^{-2}	0.62
1600	1.38×10^{-1}	0.82	5.31×10^{-2}	0.73
3200	7.50×10^{-2}	0.88	3.09×10^{-2}	0.78
6400	3.99×10^{-2}	0.91	1.79×10^{-2}	0.79

(d) L^1 error in space, second-order scheme, $T_{\max} =$ 200

N	h		hv	
200	6.16×10^{-1}		1.89×10^{-1}	
400	4.02×10^{-1}	0.61	1.35×10^{-1}	0.48
800	2.45×10^{-1}	0.71	8.88×10^{-2}	0.61
1600	1.40×10^{-1}	0.81	5.39×10^{-2}	0.72
3200	7.63×10^{-2}	0.87	3.16×10^{-2}	0.77
6400	4.06×10^{-2}	0.91	1.85×10^{-2}	0.77

Perspectives

- 2D : first and second-order schemes
- Take temperature into consideration in the system

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- 2D : first and second-order schemes
- Take temperature into consideration in the system

Thanks for your attention

Numerical results : geostrophic steady state

Table: L^1 error in space at time $T_{\max} = 200$, imposing the steady state detection.

(a) first-order scheme

N	h		hv	
200	5.25×10^{-5}		2.11×10^{-4}	
400	1.31×10^{-5}	2.00	5.30×10^{-5}	1.99
800	3.30×10^{-6}	1.99	1.38×10^{-5}	1.94
1600	8.58×10^{-7}	1.94	3.73×10^{-6}	1.88
3200	2.30×10^{-7}	1.91	1.02×10^{-6}	1.87
6400	6.01×10^{-8}	1.93	2.73×10^{-7}	1.90

(b) second-order scheme

N	h		hv	
200	5.26×10^{-5}		2.11×10^{-4}	
400	1.31×10^{-5}	2.00	5.27×10^{-5}	2.00
800	3.29×10^{-6}	2.00	1.32×10^{-5}	2.00
1600	8.22×10^{-7}	2.00	3.30×10^{-6}	2.00
3200	2.05×10^{-7}	2.00	8.25×10^{-7}	2.00
6400	5.14×10^{-8}	2.00	2.06×10^{-7}	2.00

Numerical results : transcritical flow with shock

$$h_0(x) = 0.33, hu_0(x) = 0.18, hv_0(x) = 0.$$

$$f = 0, g = 9.81.$$

$$z(x) = \begin{cases} 0.2 - 0.05(x-10)^2 & \text{if } 8 \leq x \leq 12, \\ 0 & \text{else.} \end{cases}$$

$hu = 0.18$ on the left boundary and $h = 0.33$ on the right one.